Supersymmetric deformations, RG flows and their supergravity duals

Southeastern regional mathematical string theory meeting

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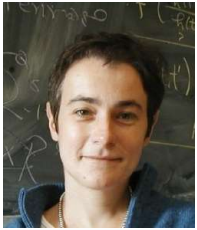
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Set-up

Focus on type IIB solutions in 10d with 4d $N = 1$ SCFT duals

- Canonical example

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Leftrightarrow \quad \text{4d } N = 4 \text{ SYM}$$

- Generalisation with all fluxes [Gauntlett et al. '05]

$$\mathrm{AdS}_5 \times M \quad \Leftrightarrow \quad \text{4d } N = 1 \text{ SCFT}$$

Few known solutions

- metric + $F_5 \Rightarrow M$ is Sasaki–Einstein [Acharya et al. '98]
- Pilch–Warner [Pilch, Warner '00]
- β deformation [Lunin, Maldacena '05]

Questions

How do we find more general solutions?

- Isolated solutions? *Hard*
- Deformations of known solutions? *This talk*

What can we understand about these backgrounds without an explicit solution?

- What information can we access? *Holomorphic data?*

Inspiration / guidance from dual field theories

4d $N = 4$ SYM in $N = 1$ language

Three chiral fields Φ^i with SU(3) global symmetry and superpotential

$$\mathcal{W} = \frac{1}{6}\epsilon_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k)$$

F-terms: $\mathrm{d}\mathcal{W} = 0 \Rightarrow [\Phi^2, \Phi^3] = 0$ so Φ^i commute

Chiral ring $\leftrightarrow$ ring of holomorphic functions on $C(S^5) = \mathbb{C}^3$:

$$\mathcal{O}_f = f_{i_1\ldots i_n}\,\mathrm{tr}(\Phi^{i_1}\ldots\Phi^{i_n}) \quad \leftrightarrow \quad f_{i_1\ldots i_n} z_{i_1}\ldots z_{i_n}$$

Hilbert series: $\mathrm{U}(1)_R$-graded count of single-trace operators modulo F-term relations

$$H(t) = \sum_k n_k t^k = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \ldots$$

Marginal deformations

e.g. $N = 1$ marginal deformations of $N = 4$ SYM [Leigh, Strassler '95]

$$\delta\mathcal{W} = f_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k)$$

where $f_{ijk} \in \mathbf{10}_{\mathbb{C}}$ of SU(3) – 10 complex couplings

One-loop beta functions

$$f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta^j_i f_{klm}\bar{f}^{klm} = 0$$

leave two exactly marginal couplings that describe conformal manifold [Kol '02, Kol '10, Green et al. '10]

$$\mathcal{M}_c = \{f_{ijk}\} /\!/ \mathrm{SU}(3) = \{f_{ijk}\} / \mathrm{SL}(3,\mathbb{C})$$

Superpotential and chirals

We can choose

$$\delta\mathcal{W} = f_\beta \operatorname{tr}\left(\Phi^1\Phi^2\Phi^3 + \Phi^3\Phi^2\Phi^1\right) + f_\lambda \operatorname{tr}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right)$$

F-term relations define non-commutative Sklyanin algebra [Ginzburg '06]

$$\tfrac{1}{2}[\Phi^1, \Phi^2] + f_\beta(\Phi^1\Phi^2 + \Phi^2\Phi^1) + 3f_\lambda(\Phi^3)^2 = 0.$$

Single-trace operators for generic (f_β, f_λ) counted by [Van den Bergh '94]

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + \dots$$

Counting not known for other marginally deformed $N = 1$ SCFTs

Dual geometries?

What do we know about the dual supersymmetric geometries? *Not much*

- $f_\lambda = 0$: β deformation preserves $U(1)^2$ isometry; exact supergravity solution known [Lunin, Maldacena '05]
- For S^5, *tour de force* 3rd-order perturbative analysis [Aharony, Kol, Yankielowicz '02], but full solution not known

Generic case: no isometries other than $U(1)_R$; *no hope?*

- But field theory seems so simple...

More questions

What kind of geometry is dual to a general 4d $N = 1$ SCFT?

- What captures the superpotential in the dual geometry?
- Can we count chiral operators in the dual geometry?

Can we find the backgrounds dual to a marginally deformed $N = 1$ SCFT?

- If not the full solution, maybe some kind of holomorphic data?
- Can we count operators around the deformed solutions?

Some answers

What kind of geometry is dual to a general 4d $N = 1$ SCFT?

- Structures (X, K) in $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ generalised geometry
- Superpotential encoded by class $[X]$
- Chiral operators $\leftrightarrow$ deformations define supersymmetric domain-wall

Can we find the backgrounds dual to a marginally deformed $N = 1$ SCFT?

- Focus on deformations of Sasaki–Einstein, e.g. S^5, $\mathrm{T}^{1,1}$, ...
- Solve for weaker "exceptional Sasaki" structure and argue existence of full solution
- Any representative of $[X]$ sufficient for counting operators

Plan of talk

1. Supersymmetry & generalised geometry
2. Counting chirals around Sasaki–Einstein
3. Exceptional Sasaki geometry
4. Counting chirals for marginally deformed SCFTs

Supersymmetry & generalised geometry

Supersymmetric AdS_5 backgrounds

General type IIB solution in 10d preserving 8 supercharges with all fields $(\Delta, \tau, H, F_3, F_5, g)$

$$ds_{10}^2 = e^{2\Delta} ds^2(AdS_5) + ds^2(M)$$

Symmetries: GDiff $\sim$ diffeos + p-form gauge

$$\delta g = \mathcal{L}_v g, \qquad \delta B^i = \mathcal{L}_v B^i - d\lambda^i, \qquad \delta C_4 = \mathcal{L}_v C_4 - d\rho + \tfrac{1}{2}\epsilon_{ij} d\lambda^i \wedge B^j$$

Supersymmetry: fermions $= 0$ and $\delta_\epsilon(\text{fermions}) = 0$

$$\nabla_m \epsilon_i + (\text{flux})_m \cdot \epsilon_i = 0, \qquad \gamma^m \nabla_m \epsilon_i + \text{flux} \cdot \epsilon_i = 0$$

Example: Sasaki–Einstein

Geometry on M defined by nowhere-vanishing tensors σ_a, j_{ab} and Ω_{ab}

- Defined by spinor bilinears: $j_{ab} \sim \bar{\epsilon}\gamma_{ab}\epsilon$, etc.
- Nowhere-vanishing vector: $\xi = g^{-1}\sigma$

Tensors define $\mathrm{SU}(2) \subset \mathrm{GL}(5, \mathbb{R})$ structure

$$\imath_\xi \sigma = 1, \qquad \imath_\xi j = \imath_\xi \Omega = j \wedge \Omega = 0, \qquad j \wedge j = \tfrac{1}{2}\Omega \wedge \bar{\Omega}$$

Supersymmetry $\Rightarrow$ differential conditions on tensors

$$\mathrm{d}\sigma = 2j, \qquad \mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega, \qquad F_5 = 4\,\mathrm{vol}_M$$

where ξ is a Killing vector, dual to $\mathrm{U}(1)_R$ of SCFT, that preserves full solution

SUSY backgrounds with flux

Long history of using G-structures and generalised geometry to analyse supersymmetric backgrounds [Hitchin '03; Gualtieri '11; Hull '07; Pacheco, Waldram '08; Coimbra, Strickland-Constable, Waldram '11; Berman et al. '11; Anderson, Gray, Sharpe '14;...]

- General AdS_5 case: spinor $\epsilon = (\epsilon_1, \epsilon_2)$ stabilised by $USp(6)$ [Coimbra, Strickland-Constable, Waldram '14]

Defined by pair (X, K) in $E_{6(6)} \times \mathbb{R}^+$ generalised geometry

$$X \sim \text{hypermultiplet d.o.f.} \qquad K \sim \text{vector multiplet d.o.f.}$$

Generalised tensors transform as reps of $E_{6(6)} \times \mathbb{R}^+$

- Sums of conventional tensors under $GL(5, \mathbb{R}) \subset E_{6(6)} \times \mathbb{R}^+$

K structure

Generalised vector V parametrises diffeo + gauge transformations ($T \equiv TM$)

$$\mathbf{27} \sim E \simeq T \oplus 2T^* \oplus \Lambda^3 T^* \oplus 2\Lambda^5 T^*$$
$$V^A = v^a + \lambda_a^i + \rho_{abc} + \sigma^i_{abcde}$$

Symmetric cubic form c on E

$$c(V,V,V) = -\tfrac{1}{2} \imath_v \rho \wedge \rho + \ldots \quad \in \det T^*$$

Vector-multiplet structure K defined by

$$K \in \Gamma(E) \quad \text{s.t.} \quad c(K,K,K) > 0$$

K defines an $\mathrm{F}_{4(4)}$ structure

X structure

Weighted adjoint bundle

$$\det T^* \otimes \mathbf{78} \sim T^* \oplus 2\Lambda^3 T^* \oplus \dots$$
$$R^A{}_B = \eta_{Ia} + \Sigma^i_{abc} + \dots$$

Hypermultiplet structure X defined by

$$\mathrm{tr}(X\bar{X}) = -2\kappa^4 < 0 \qquad \text{where } \kappa^2 \in \det T^*$$

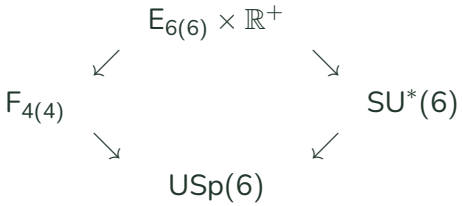
where X defines a highest-root $\mathfrak{su}_2$ algebra and a third generator J

$$J = \frac{\mathrm{i}}{4\kappa^4}[X, \bar{X}]$$

X defines an SU*(6) structure

A compatible pair

Spinor ϵ defines the pair (X, K)



Intersect on USp(6) if compatible

$$X \cdot K = 0$$

(X, K) specify all supergravity fields for solution

Example: Sasaki–Einstein

Recall SU(2) structure defined by tensors (ξ, σ, j, Ω)

K encodes a contact structure on M

$$K = \xi - \sigma \wedge j \quad \in T \oplus \Lambda^3 T^*$$

X encodes a Cauchy–Riemann structure on M

$$X = n^i \sigma \wedge \Omega \quad \in 2\Lambda^3 T^*$$

with axio-dilaton τ

$$n^i = \frac{1}{\sqrt{\operatorname{im} \tau}} (1, \tau)^i$$

Geometry on space of structures

Space of X's is hyper-Kähler cone over quaternionic-Kähler space

Infinitesimally, $V \in \Gamma(E) \simeq \mathfrak{gDiff}$ acts by Dorfman derivative

$$\delta X = L_V X, \qquad L_V = \mathcal{L}_v - (\mathrm{d}\lambda^i + \mathrm{d}\rho)\cdot \sim \text{diffeo} + \text{gauge}$$

Preserves HK structure so gives a triplet of moment maps for GDiff

Naturally written as a holomorphic moment map μ_+ and a real moment map μ_3

$$\mu_+(V) = -\mathrm{i}\int_M \mathrm{tr}(J L_V X), \qquad \mu_3(V) = -\frac{\mathrm{i}}{2}\int_M \kappa^{-2}\,\mathrm{tr}(X L_V \bar{X})$$

Supersymmetry of the solution is equivalent to

$$L_K K = 0, \qquad L_K X = 3\mathrm{i}X,$$
$$\mu_+(V) = 0, \qquad \mu_3(V) = 3\int_M c(K,K,V) \quad \forall\, V$$

"Exceptional Sasaki–Einstein" – generalised USp(6) structure with constant singlet torsion [AA, Petrini, Waldram '16]

- K is generalised Killing vector – encodes $\mathrm{U}(1)_R$ of SCFT
- X is charge three

Example: Sasaki–Einstein

Each recovers some of the SUSY conditions

$$L_K K = 0 \qquad \Rightarrow \qquad \mathrm{d}j = 0$$

$$L_K X = 3\mathrm{i}X \qquad \Rightarrow \qquad \mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega$$

$$\mu_+(V) = 0 \qquad \Rightarrow \qquad \mathrm{d}(\sigma \wedge \Omega) = 0$$

$$\mu_3(V) = 3\int_M \mathrm{c}(K, K, V) \qquad \Rightarrow \qquad \mathrm{d}\sigma = 2j, \quad F_5 = 4\,\mathrm{vol}_M$$

Counting chirals around Sasaki–Einstein

Counting chiral operators

Chiral ring defined by chiral operators modulo F-term relations

$$\text{chirals} = \{\text{chiral operators}\} \,/\, \{\mathrm{d}\mathcal{W} = 0\}$$

Superpotential deformations preserve $N = 1$, but not necessarily conformal

- Can trigger RG flow
- Holomorphic RG scheme $\Rightarrow$ $\mathcal{W}$ not renormalised [Seiberg '93]

How do we see this in the dual geometry?

- RG flow $\leftrightarrow$ supersymmetric domain-wall solution
- Identify dual of F-term conditions and holomorphic data preserved along flow

Supersymmetric domain-wall solution

Replace AdS_5 with domain-wall metric

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2$$

where $A = A(r)$

- Dual to flowing $N = 1$ QFT
- X and K now depend on r

5d gauged supergravity coupled to ∞-many hyper and vector multiplets [de Wit, Nicolai '85, '86]

- Hyper / vector scalar manifolds $\rightarrow$ spaces of X and K
- BPS conditions $\rightarrow$ constraint and flow equations for X and K

Lifting the BPS conditions

BPS conditions given by [Ceresole et al. '01] in terms of quaternionic-Kähler and very special real scalar manifolds

- Hides "nice" geometry – move to superconformal cones over each space
- Scalar manifolds become hyper-Kähler cone and very special real cone

Superconformal cones embed in spaces of X and K

- Gauging lifts to Dorfman derivative $L_V K$ and $L_V X$
- Moment maps for gauging lift to $\mu_\alpha(V)$

Domain-wall equations

Supersymmetry of domain wall equivalent to

$$\mu_+(V) = 0, \qquad L_K K = 0,$$

$$L_K X = \mathrm{i}\dot{X}, \qquad \mu_3(V) = 3 \int_M c(K, \dot{K}, V).$$

First line: supersymmetry constraints – holomorphic in X

Second line: RG flow – X flows by imaginary GDiff

- Flow preserves compatibility of (X, K) and constraints

Holomorphic data and RG flow

Chiral operators $\leftrightarrow$ hypers $\leftrightarrow$ X

- SCFT: supersymmetric deformations not necessarily conformal
- Geometry: holomorphic deformations of X that solve $\mu_+ = 0$

What is the holomorphic data?

- SCFT: $\mathcal{W}$ not renormalised
- Geometry: X preserved up to $\mathrm{GDiff}^{\mathbb{C}}$

Class of X preserved along flow [AA, Petrini, Tasker, Waldram '21]

$$[X] = \{X \sim \phi^*(X) \mid \mu_+ = 0,\ \phi \in \mathrm{GDiff}^{\mathbb{C}}\}$$

Quantities determined solely by $\mathcal{W}$ can be computed using any representative of $[X]$

Counting chiral operators

Field theory: chiral operators modulo F-terms

$$\text{chirals} = \{\text{chiral operators}\} \,/\, \{\mathrm{d}\mathcal{W} = 0\}$$

Geometry: infinitesimal holomorphic deformations

$$\text{chirals} = \{\delta X \mid \delta X \cdot K = 0,\ \delta\mu_+ = 0\} \,/\, \mathrm{GDiff}^{\mathbb{C}}$$

Depends only on class of $[X]$

- Counts relevant, marginal and irrelevant deformations
- Graded by charge under $\mathrm{U}(1)_R$ or L_K

Example: Sasaki–Einstein

We find a family of linearised solutions to $\mu_+ = 0$

$$K = \xi - \sigma \wedge j, \qquad X_*(f) = \mathrm{d}f + n^i \sigma \wedge \Omega$$

where f is a charge-k holomorphic function

For S^5, $f = f_{i_1 \ldots i_k} z_{i_1} \ldots z_{i_k}$, so Hilbert series counts charge-k polynomials in (z_1, z_2, z_3):

$$H(t) = \sum_k n_k t^k = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \ldots$$

- Matches Hilbert series of $N = 4$ SYM theory ✓
- General SE counted by Kohn–Rossi cohomology [Eager, Schmude, Tachikawa '12]

Exceptional Sasaki geometry

General solutions

Can we find supergravity solutions dual to marginally deformed field theories?

- Solving for full solutions seems intractable – no isometries; harder than Calabi–Yau
- Can we still calculate *something*?

Focus on subset of supersymmetry conditions

$$L_K K = 0, \qquad L_K X = 3\mathrm{i}X,$$
$$\mu_+(V) = 0, \qquad \mu_3(V) = 3\int_M c(K, K, V) \quad \forall\, V$$

Defines "exceptional Sasaki" geometry

Moduli space

$\mathcal{Z}_K$ is space of X which are exceptional Sasaki for *fixed* K

$$\mathcal{Z}_K = \{X \mid \mu_+ = 0,\, L_K K = 0,\, L_K X = 3\mathrm{i}X\}$$

Final supersymmetry condition is a real moment map for GDiff_K

$$\mu_K(V) \coloneqq \mu_3(V) - 3\int c(K, K, V)$$

with moduli space

$$\mathcal{M}_c \simeq \frac{\{X \in \mathcal{Z}_K \;\mid\; \mu_K = 0\}}{\mathrm{GDiff}_K} \equiv \mathcal{Z}_K /\!/ \mathrm{GDiff}_K$$

$\mathcal{M}_c$ is conformal manifold of dual SCFT [Kol '02, Kol '10, Green et al. '10]

GIT quotient

$\mathcal{Z}_K$ admits a GDiff_K-invariant Kähler metric so equivalent to quotient by complexified group

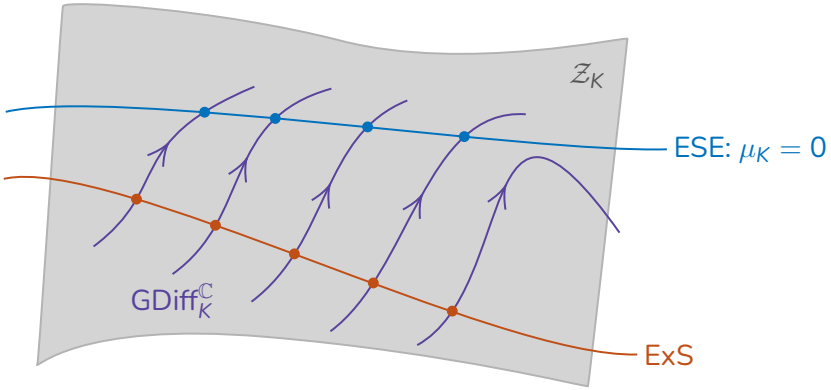
$$\mathcal{M}_{\mathrm{c}} \simeq \mathcal{Z}_K \,/\!/\, \mathrm{GDiff}_K \simeq \mathcal{Z}_K^{\mathrm{p.s.}} \,/\, \mathrm{GDiff}_K^{\mathbb{C}}$$

i.e. final supersymmetry condition is a GIT quotient

- Favourable case: orbit of X under $\mathrm{GDiff}_K^{\mathbb{C}}$ intersects with $\{\mu_K = 0\}$

Useful: can find simpler exceptional Sasaki solution in same class, then argue it can be completed to full solution

Stability and existence



Subtlety: only polystable X's have orbits that reach $\{\mu_K = 0\}$

Strategy

1. Start with known solution (X, K)
2. Deform to (X_*, K) which solves weaker exceptional Sasaki conditions
3. Argue that X_* is polystable so can be $\mathrm{GDiff}_K^{\mathbb{C}}$-transformed to a full supersymmetric solution
4. Study the dual deformed field theory – anything that depends only on $[X_*]$ can be calculated, e.g. chiral spectrum

This is a general picture, but we'll focus on cases where the starting solution is Sasaki–Einstein

Deformations of Sasaki–Einstein

We can extend our linearised solutions to all orders

$$K = \xi - \sigma \wedge j, \qquad X_*(f) = \mathrm{e}^{b^i(f)}\big(\mathrm{d}f + v^i(f)\,\sigma \wedge \Omega\big)$$

where

- f is a charge-three holomorphic function $\Rightarrow$ marginal deformation
- $b^i(f)$ linear and $v^i(f)$ quadratic in f

Solves the exceptional Sasaki conditions

- Can these be completed to full solutions?

Can they be completed?

For these exceptional Sasaki solutions we know:

1. $X_*(f)$ gives continuous one-parameter family of solutions
2. $X_*(f)$ matches known infinitesimal solutions to $\mu_K = 0$ [AA, Gabella, Graña, Petrini, Waldram '16]
3. Moment map $\Rightarrow$ unobstructed after accounting for residual symmetries

So for allowed f, $X_*(f)$ are polystable and can be completed to full supersymmetric solutions

- For S^5 and $f = f_\beta\, z_1 z_2 z_3$, can find $\mathrm{GDiff}_K^{\mathbb{C}}$ that reproduces β-deformed solution of [Lunin, Maldacena '05]

Counting chirals for marginally deformed SCFTs

Chiral spectrum around X_*

Chirals of deformed SCFT $\leftrightarrow$ infinitesimal deformations of X_*

- Field theory: non-commutative algebra $\mathrm{d}\mathcal{W} = 0$
- Geometry: linearised holomorphic deformations $\delta\mu_+ = 0$

Can use any representative of class, e.g. X_*

$$\text{chirals} = \{\delta X_* \mid \delta X_* \cdot K = 0,\ \delta\mu_+ = 0\} \,/\, \mathrm{GDiff}^{\mathbb{C}}$$

Counting can again be graded by charge under L_K

Sasaki–Einstein and marginally deformed SCFTs

Generic deformation: $\eta := \mathrm{d}f$ nowhere vanishing

- Counting reduces to computing η cohomology [Tasker '21]

$$\ldots \xrightarrow{\mathrm{d}} \eta \wedge \Lambda^p T^* \xrightarrow{\mathrm{d}} \eta \wedge \Lambda^{p+1} T^* \xrightarrow{\mathrm{d}} \ldots$$

- Computed by Kohn–Rossi cohomology of original Sasaki–Einstein

Universal expression for Hilbert series of marginally deformed SCFTs dual to deformed Sasaki–Einstein

$$\boxed{H(t) = 1 + \mathcal{I}_{\text{s.t.}}(t^{1/2}) - \frac{t^3}{1-t^3}}$$

where $\mathcal{I}_{\text{s.t.}}$ is single-trace superconformal index of undeformed theory [Eager, Schmude, Tachikawa '12]

Example: marginally deformed $N = 4$ SYM

Marginally deformed $N = 4$ SYM dual to $AdS_5 \times S^5$ deformed by

$$f = f_\beta \, z_1 z_2 z_3 + f_\lambda \left((z_1)^3 + (z_2)^3 + (z_3)^3 \right)$$

Hilbert series given by

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + 3t^4 + \dots$$

in agreement with counting of [Van den Bergh '94] ✓

No previous calculation of cyclic homology for other deformed theories

- New results for all regular SE's: $T^{1,1}$, $\#n(S^2 \times S^3)$, etc.

Summary

Generalised geometry naturally packages information about dual field theory

- X and K encode hyper and vector d.o.f.

RG flow of field theory becomes domain-wall flow of X and K

- Flow preserves class $[X]$, $\delta\mu_+ = 0$ is F-term condition

Can find supergravity backgrounds dual to marginal deformations up to $\mathrm{GDiff}_K^{\mathbb{C}}$

- Further deformations count chirals without needing full solution

Outlook

AdS_4/ AdS_5 in type II / M-theory

Supersymmetric indices for flux backgrounds

a-maximisation and stability

Appendix

Example: $T^{1,1}$

e.g. Klebanov–Witten dual to $AdS_5 \times T^{1,1}$ has Hilbert series

$$H(t) = \frac{1 + t^{3/2}}{(1 - t^{3/2})^3} = 1 + 4t^{3/2} + 9t^3 + 16t^{9/2} \dots$$

For generic superpotential deformation

$$H(t) = \frac{1 + 4t^{3/2} + 2t^3}{1 - t^3} = 1 + 4t^{3/2} + 3t^3 + 4t^{9/2} + \dots$$

- No previous calculation of cyclic homology for deformed theory
- Matches explicit counting of gauge-invariant chiral fields modulo F-term relations up to $k = 21/3$ [Tasker '21]