SKIDMORE COLLEGE
SKIDMORE COLLEGE
DEPARTMENT OF PHYSICS

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PHASES OF LATTICE GAUGE THEORIES VIA NEURAL NETWORKS

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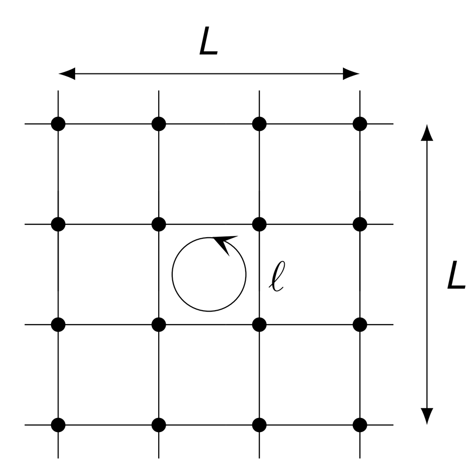
GOAL

To locate and characterize the transition between the **confined** and **deconfined** phases in (2+1)D $\mathbb{Z}_N$ lattice gauge theories for $N = 4, 5, 6$ using gauge-equivariant neural quantum states following the approach proposed by Apte et al. [1]. Here, we focus on $\mathbb{Z}_4$.

LATTICE GAUGE THEORY

$\mathbb{Z}_N$ lattice gauge theory is a simplified quantum mechanical model of **field forces** like electromagnetism and the strong nuclear force. Space is discretized as an **$L \times L$ lattice** while time remains continuous.

The field lives on the links between sites. Each link on the lattice has N basis states: $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$. The state $|\psi\rangle$ of the full system is a linear combination of tensor products of these basis states.



A 4×4 lattice. States live on the links ℓ between sites.

The theory is described by unitary operators P and Q which act on the states:

$$Q_\ell |q\rangle_\ell = e^{2\pi i q/N} |q\rangle_\ell, \qquad P_\ell |q\rangle_\ell = |q-1 \bmod N\rangle_\ell.$$

The physical behavior is captured by a **Hamiltonian** H:

$$H = \frac{g^2}{2} \sum_l \left[2 - (P_l + P_l^\dagger)\right] + \frac{1}{2g^2} \sum_{p=(l_1,l_2,l_3,l_4)} \left[2 - (Q_{l_1}^\dagger Q_{l_2}^\dagger Q_{l_3} Q_{l_4} + \text{h.c.})\right] = H_E + H_B.$$

The first term is the **electric field** energy, the second is the **magnetic field** energy. The **coupling g** changes the relative importance of these terms.

We want to find the **ground state** $|\psi_0\rangle$ of this system – the lowest energy state.

Exact diagonalization works only for very small systems because the size of the Hamiltonian grows exponentially. For a 6×6 lattice, it requires diagonalizing a matrix of size $N^{72} \times N^{72}$.

Gauge invariance: The physical fields are the electric and magnetic fields, but the Hamiltonian is built from potentials. This redundancy gives the theory a local **$\mathbb{Z}_N$ gauge symmetry**. The true ground state $|\psi_0\rangle$ is **invariant** under the gauge symmetry – this is the quantum version of Gauss' law.

PHASE TRANSITIONS

This theory undergoes a **phase transition** at a critical coupling g_c.

Deconfined phase $g < g_c$: Electric field lines spread throughout the lattice. Similar to electromagnetism, charges are essentially free in this phase.

Confined phase $g > g_c$: Electric field lines are tightly bound in a flux tube connecting two charges. The potential between two charges grows linearly with separation r:

$$\boxed{V(r) = \sigma r}$$

where the coefficient σ is known as the **string tension**.

NEURAL NETWORKS AND NEURAL QUANTUM STATES

A **neural network** is a flexible nonlinear function with a large number of tunable parameters. Neural networks have recently been shown to be powerful tools in finding ground states of complex many-body quantum systems such as spin systems. These **neural quantum states** (NQSs) use neural networks to act as a variational ansatz (a guess with parameters) for the quantum wavefunction [2].

NQSs use **Monte Carlo sampling** via Markov chains and the **variational principle** to find the ground-state wavefunction ψ_0.

- The wavefunction ψ_θ is modeled by a neural network with parameters (weights) θ.
- The network is trained using **gradient descent** to update θ to minimize the energy $\langle H \rangle$.
- After training, the NQS ψ_θ will approximate the true ground-state wavefunction ψ_0.

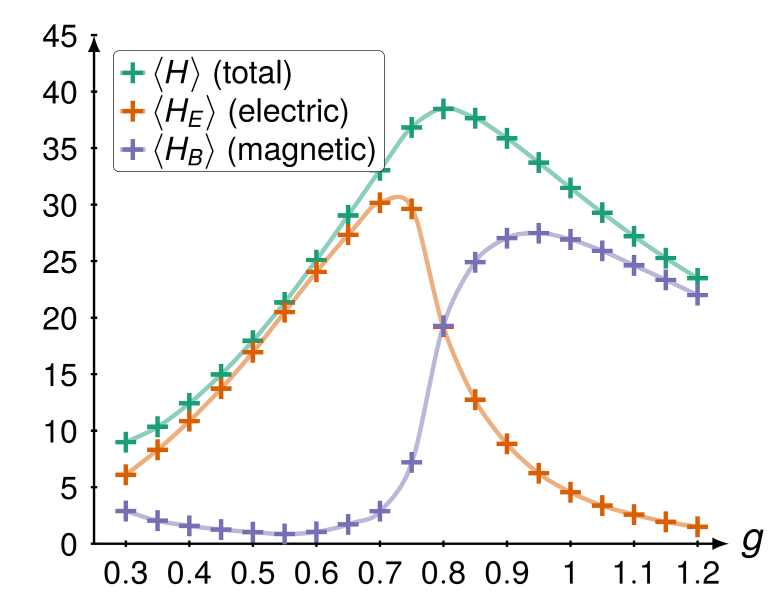
The wavefunction is modeled by a nonlinear function with a polynomial number of parameters (the network's weights) instead of storing an exponentially large number of probability amplitudes. This avoids the issue that prevents exact diagonalization.

NETWORK STRUCTURE AND GROUND-STATE ENERGY

We use a **lattice gauge equivariant convolutional neural network** (LGE-CNN), an architecture first developed by Favoni et al. [3], and then used by Apte et al. [1] to study $\mathbb{Z}_2$ and $\mathbb{Z}_3$ theories. We constructed and trained our networks using the Python package **NetKet** [4].

As the ground state is gauge invariant, the neural network should also be gauge invariant. LGE-CNNs are useful as they are gauge invariant by construction.

Once trained, the NQS can be used to compute many physical quantities of interest, including the **ground-state energy** of the system as g is varied, shown to the right.

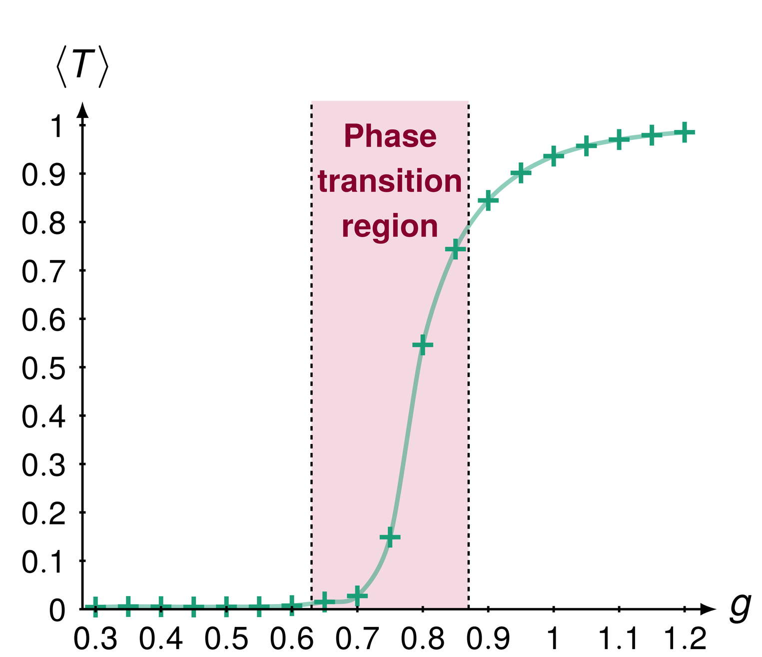


Total, electric and magnetic energy of the ground state of a $\mathbb{Z}_4$ theory on a 6×6 lattice. $\langle H_E \rangle$ and $\langle H_B \rangle$ exchange dominance near g_c.

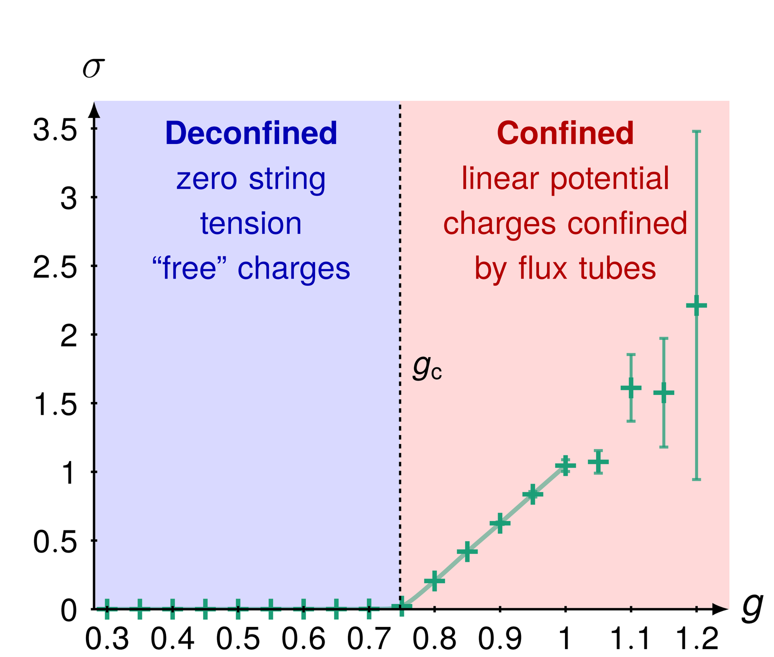
DETECTING THE PHASE TRANSITION

't Hooft string $\langle T \rangle$: A 't Hooft string creates a pair of magnetic monopoles in the system. Its expectation value behaves differently in the confined and deconfined phases, so it can be used to detect the phase transition.

String tension σ: The string tension acts as an **order parameter**, and can distinguish between the phases. The string tension should be zero in the deconfined phase and become positive in the confined phase. The crossover gives an estimate of the region where the phase transition occurs.



't Hooft string expectation value $\langle T \rangle$ for $\mathbb{Z}_4$ on a 6×6 lattice as a function of coupling g. It is zero in the deconfined phase and tends to one in the confined phase. The crossover region is where the phase transition occurs.



String tension σ for $\mathbb{Z}_4$ on a 6×6 lattice as a function of coupling g. The increase in the string tension gives an estimate of the critical coupling g_c. Error bars grow with coupling as confinement exponentially suppresses the Wilson loops used in the calculation, amplifying statistical noise.

CRITICAL EXPONENTS

Phase transitions can be characterized by numbers called **critical exponents**. We can use our NQSs to compute these exponents β and ν for the $\mathbb{Z}_4$ theory.

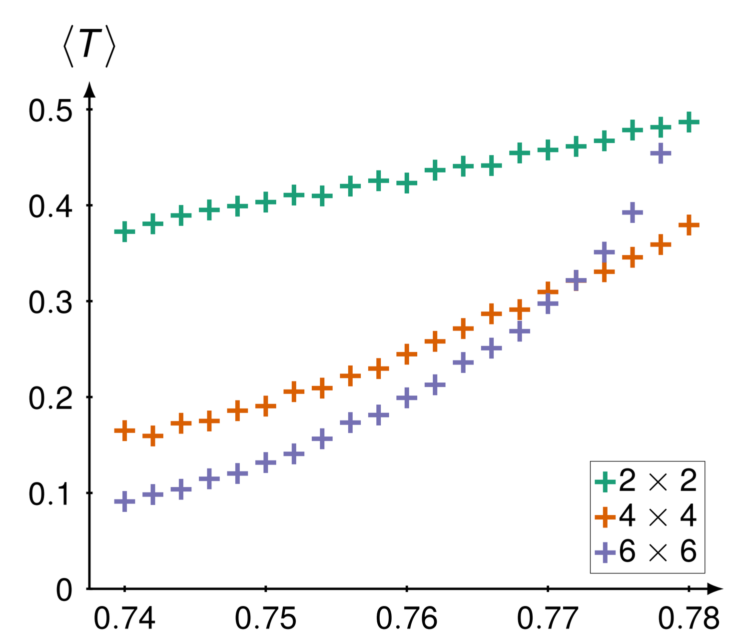
Finite-size scaling: True phase transitions exist only for infinite systems. Near the transition, the **correlation length** ξ diverges as a power of $1/|g - g_c|$, while physical quantities depend only on the dimensionless ratio ξ/L. Using this, we can extract the critical exponents.

Finite-size scaling predicts that if g_c, β and ν are chosen correctly, the curves of $L^{2\beta/\nu}\langle T \rangle$ against $L^{1/\nu}(g - g_c)$ will be independent of the lattice size and **collapse** onto a single curve.

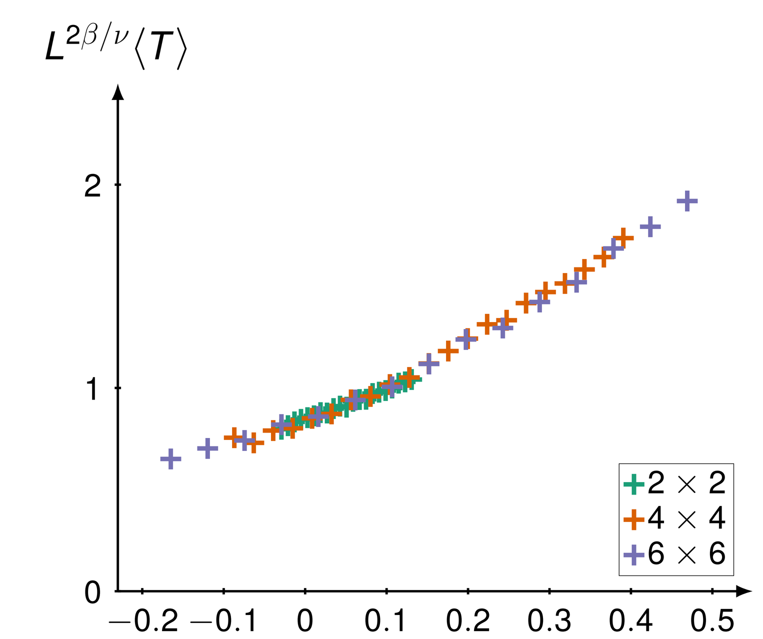
We determined g_c and the critical exponents via **curve collapse** to be

$$\boxed{g_c = 0.7473(5), \quad \beta = 0.347(7), \quad \nu = 0.63(1).}$$

Below we show $\langle T \rangle$ for three lattice sizes near the phase transition, and show that the three curves collapse for our calculated critical exponents.



$\langle T \rangle$ for $\mathbb{Z}_4$ on $L = 2, 4, 6$ lattices.



Collapsed curves for $\mathbb{Z}_4$ on $L = 2, 4, 6$ lattices.

The $\mathbb{Z}_4$ theory factorizes into two independent $\mathbb{Z}_2$ theories. We therefore expect its critical exponents to match those of the **3d Ising model**.

- Previous calculations give $g_c = 0.757051$, $\beta = 0.326419(3)$ and $\nu = 0.629971(4)$ [5, 6].
- Our results match to within 1.3% for g_c, 6.5% for β and 0.55% for ν.

These results are encouraging given that we used small lattices and nonlinear curve fitting, and suggest that neural networks are a promising way to model $\mathbb{Z}_N$ theories beyond the previously studied $\mathbb{Z}_2$ and $\mathbb{Z}_3$ theories.

FUTURE DIRECTIONS

- Extend calculations to **larger lattices** to improve accuracy of critical exponent estimates.
- Apply NQSs to $\mathbb{Z}_5$ and $\mathbb{Z}_6$ theories which are believed to have **two** phase transitions.
- Explore how NQSs can be used in SU(3) theories which directly model the **strong force**.

References

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