Exactly marginal deformations and their supergravity duals

Anthony Ashmore

Sorbonne Université

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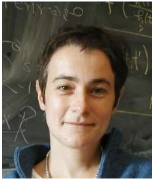
Collaborators

Dan Waldram



Michela Petrini

Ed Tasker

Stephanie Baines

Motivation & “The Question”

Set-up

Focus on type IIB solutions in 10d with 4d $N = 1$ SCFTs duals

- Canonical example

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \quad \Leftrightarrow \quad 4\mathrm{d}\ N = 4\ \mathrm{SYM}$$

- Generalisation with all fluxes

$$\mathrm{AdS}_5 \times M \quad \Leftrightarrow \quad 4\mathrm{d}\ N = 1\ \mathrm{SCFT}$$

Known solutions

- e.g. metric + $F_5 \Rightarrow M$ is Sasaki–Einstein
- Extra isometries: e.g. Pilch–Warner [Pilch, Warner '00]; β deformation [Lunin, Maldacena '05]

Questions

How do we find more general solutions?

- Isolated solutions? *Hard*
- Deformations of known solutions? *This talk*

What can we understand about these solutions without their explicit form?

- What information might we have access to?

Inspiration / guidance from dual field theories

4d $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ language

Three chiral fields Φ^i with SU(3) global symmetry and superpotential

$$\mathcal{W} = \epsilon_{ijk} \operatorname{tr}(\Phi^i \Phi^j \Phi^k)$$

F-terms $d\mathcal{W} = 0 \Rightarrow \Phi^i$ commute: $[\Phi^2, \Phi^3] = 0$

Chiral ring $\leftrightarrow$ ring of holomorphic functions on $C(S^5) = \mathbb{C}^3$:

$$\mathcal{O}_f = f_{i_1 \ldots i_n} \operatorname{tr}(\Phi^{i_1} \ldots \Phi^{i_n}) \quad \leftrightarrow \quad f_{i_1 \ldots i_n} z^{i_1} \ldots z^{i_n}$$

Hilbert series: graded count of single-trace operators modulo F-term relations

$$H(t) = \sum_k n_k t^k = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \ldots$$

Marginal deformations

e.g. $N = 1$ deformations of $N = 4$ SYM [Leigh, Strassler '95]

$$\delta\mathcal{W} = f_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k)$$

- $f_{ijk} \in \underline{10}_{\mathbb{C}}$ of SU(3) – 10 complex d.o.f.
- One-loop beta functions

$$f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i^j f_{klm}\bar{f}^{klm} = 0$$

$2_{\mathbb{C}}$ exactly marginal couplings give conformal manifold [Kol '02, Kol '10, Green et al. '10]

$$\mathcal{M}_{\mathrm{c}} = \{f_{ijk}\} \,//\, \mathrm{SU}(3) = \{f_{ijk}\} \,/\, \mathrm{SL}(3,\mathbb{C})$$

Superpotential and chirals

We can choose

$$\delta\mathcal{W} = f_\beta \, \mathrm{tr}(\Phi^1\Phi^2\Phi^3) + f_\lambda \, \mathrm{tr}\left[(\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right]$$

F-term relations define non-commutative Sklyanin algebra [Ginzburg '06]

Chiral operators for generic f_β and f_λ counted by [Van den Bergh '94]

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + \dots$$

- Counting not known for other $N = 1$ SCFTs

Dual geometries?

What do we know about the dual supersymmetric geometries?
Not much

- $f_\lambda = 0$: "β deformation", preserves $U(1)^2$ isometry, exact supergravity solution known [Lunin, Maldacena '05]
- Generic case: no isometries other than $U(1)_R$ (and no hope?)
- For S^5, *tour de force* 3rd-order perturbative analysis [Aharony, Kol, Yankielowicz '02], but full solution not known

Dual geometries?

Understand the geometry dual to a generic $N = 1$ SCFT?

- If not the full geometry, maybe some partial data?

Count the chiral operators for the deformed theories from the geometry?

- Want to count these around the deformed solutions
- Akin to counting Kaluza–Klein modes – even for explicitly known solutions, this is hard...

But field theory seems so simple?

Dual geometries?

Understand the geometry dual to a generic $N = 1$ SCFT?

- Integrable structures (X, K) in $E_{6(6)} \times \mathbb{R}^+$ gen. geometry
- Data of superpotential $\mathcal{W}$ encoded by class $[X]$

Focus on those obtained as deformations of Sasaki–Einstein, e.g. S^5, $T^{1,1}$, ...

- X solves weaker "exceptional Sasaki" conditions
- Only "holomorphic" data is explicit, but can argue for existence of full solution

Count the chiral operators for the deformed theories from the geometry?

- Class $[X]$ is sufficient to calculate holomorphic quantities, e.g. Hilbert series

Plan of talk

1. Review supersymmetry via generalised structures
2. Describe supergravity analogue of holomorphic data encoded by $\mathcal{W}$
3. Give holomorphic data that determines full solution up to complexified diffeos + gauge
4. Compute chiral spectrum for deformed SCFTs from dual geometry

Supersymmetry & generalised structures

Supersymmetric AdS_5 backgrounds

Generic type IIB solution in 10d preserving 8 supercharges with all fields $(\Delta, \tau, H, F_3, F_5, g)$

$$\mathrm{d}s^2_{10} = e^{2\Delta}\mathrm{d}s^2(\mathrm{AdS}_5) + \mathrm{d}s^2(M)$$

Symmetries: GDiff $\sim$ diffeos $+$ p-form gauge

$$\delta B^i = \mathrm{d}\lambda^i, \qquad \delta C_4 = \mathrm{d}\rho - \tfrac{1}{2}\epsilon_{ij}\mathrm{d}\lambda^i \wedge \mathrm{d}B^j$$

Supersymmetry: fermions $= 0$ and $\delta_\epsilon(\text{fermions}) = 0$

$$\nabla_m \epsilon + (\text{flux})_m \cdot \epsilon = 0, \qquad \gamma^m \nabla_m \epsilon + \text{flux} \cdot \epsilon = 0$$

with $\epsilon = (\epsilon_1, \epsilon_2)$ stablised by USp(6) [Coimbra, Strickland-Constable, Waldram '14]

Example: Sasaki–Einstein

e.g. M is Sasaki–Einstein

Geometry defined by nowhere-vanishing tensors σ_m, j_{mn} and Ω_{mn}

- Defined by spinor bilinears: $j_{mn} \sim \bar{\epsilon}\gamma_{mn}\epsilon$, etc.
- Nowhere-vanishing vector $\xi = g^{-1}\sigma$

Tensors satisfy algebraic conditions

$$\imath_\xi \sigma = 1, \qquad \imath_\xi j = \imath_\xi \Omega = j \wedge \Omega = 0, \qquad j \wedge j = \tfrac{1}{2}\Omega \wedge \bar{\Omega}$$

Invariant under $\mathrm{SU}(2) \subset \mathrm{GL}(5, \mathbb{R})$

Example: Sasaki–Einstein

Supersymmetry implies differential conditions on invariant tensors

$$\mathrm{d}\sigma = 2j, \qquad \mathrm{d}\Omega = 3\mathrm{i}\sigma \wedge \Omega,$$
$$F_5 = \mathrm{d}C_4 = 4\,\mathrm{vol}_g$$

ξ is a Killing vector (Reeb), $\mathcal{L}_\xi$ preserves full solution

- ξ dual to $\mathrm{U}(1)_R$ R-symmetry of $N = 1$ SCFT
- Supersymmetry $\Rightarrow$ equations of motion

SUSY backgrounds with flux

Long history of using G-structures and generalised geometry to analyse supersymmetric flux backgrounds

Generic AdS_5 case: spinor ϵ defines integrable USp(6) structure – "exceptional Sasaki–Einstein" [AA, Petrini, Waldram '16]

- Defined by pair (X, K) in $E_{6(6)} \times \mathbb{R}^+$ generalised geometry

$$X \sim \text{hyper d.o.f.} \qquad K \sim \text{vector d.o.f.}$$

Construct gen. tensors as reps of $E_{6(6)} \times \mathbb{R}^+$

$$GL(5, \mathbb{R}) \subset E_{6(6)} \times \mathbb{R}^+$$

K structure

Generalised vector V^A parametrises diffeos + gauge transformations ($T \equiv TM$, etc.)

$$\underline{27} \sim E \simeq T \oplus 2T^* \oplus \Lambda^3 T^* \oplus 2\Lambda^5 T^*$$
$$V^A = v^a + \lambda^i_a + \rho_{abc} + \sigma^i{}_{abcde}$$

Invariant cubic form c on E

$$c(V,V,V) = -\tfrac{1}{2}\imath_v \rho \wedge \rho + \dots \quad \in \det T^*$$

K structure ("vector-multiplet" structure) defined by

$$K \in E \quad \text{s.t.} \quad c(K,K,K) > 0$$

- Generalised vector invariant under $F_{4(4)}$

X structure

Adjoint bundle

$$\underline{78} \sim \mathrm{ad}\, F \simeq 3\mathbb{R} \oplus (T \otimes T^*) \oplus 2\Lambda^2 T^* \oplus 2\Lambda^2 T \oplus \Lambda^4 T^* \oplus \Lambda^4 T$$
$$R^A{}_B = \dots + B^i_{ab} + \dots + C_{abcd} + \dots$$

X structure ("hypermultiplet" structure) defined by

$$X \in \mathrm{ad}\, F_{\mathbb{C}} \otimes \det T^* \quad \text{s.t.} \quad \mathrm{tr}(X\bar{X}) < 0$$

where $X = \kappa(J_1 + \mathrm{i}J_2)$ defines $\mathfrak{su}_2$ algebra

$$[J_\alpha, J_\beta] = 2\kappa\epsilon_{\alpha\beta\gamma}J_\gamma, \qquad \mathrm{tr}(J_\alpha J_\beta) = -\kappa^2\delta_{\alpha\beta}, \qquad \kappa^2 \in \det T^*$$

- Complex adjoint tensor invariant under $SU^*(6)$

Generalised structures

Spinor ϵ defines the pair (X, K)

$$
\begin{array}{ccccc}
 & & \mathrm{E}_{6(6)} \times \mathbb{R}^+ & & \\
 & \swarrow & & \searrow & \\
(K \in E) \quad \mathrm{F}_{4(4)} & & & & \mathrm{SU}^*(6) \quad (X \in \det T^* \otimes \mathrm{ad}_{\mathbb{C}}) \\
 & \searrow & & \swarrow & \\
 & & \mathrm{USp}(6) & & \\
 & & (X, K) & &
\end{array}
$$

Intersect on USp(6) if compatible

$$X \cdot K = 0$$

(X, K) specify all supergravity fields for solution

Example: Sasaki–Einstein

Recall G-structure defined by (σ, j, Ω)

K structure defines "contact structure"

$$K_{\mathrm{SE}} = \mathrm{e}^{C_4}(\xi - \sigma \wedge j) \quad \in T \oplus \Lambda^3 T^* \subset E$$

X structure defines "Cauchy–Riemann structure"

$$X_{\mathrm{SE}} = \mathrm{e}^{C_4 - \frac{1}{4}\mathrm{i}\,\Omega\wedge\bar{\Omega}} n^i\, \sigma \wedge \Omega \quad \in 2\Lambda^3 T^* \subset \mathrm{ad}\, F_{\mathbb{C}} \otimes \det T^*$$

where

$$n^i = \frac{1}{\sqrt{\mathrm{im}\,\tau}}(1, \tau)^i$$

Supersymmetry

Symmetries act via Dorfman derivative

$$\mathbb{L}_V = \mathcal{L}_v - (\mathrm{d}\lambda^i + \mathrm{d}\rho)\cdot \ \sim \text{diffeo} + \text{gauge}$$

Supersymmetry of the solution is equivalent to [AA, Petrini, Waldram '16]

$$\mathbb{L}_K K = 0, \qquad \mathbb{L}_K X = 3\mathrm{i}X,$$
$$\mu_+(V) = 0, \qquad \mu_3(V) = 3\int_M c(K,K,V) \quad \forall\, V$$

- "Exceptional Sasaki–Einstein" (ESE)
- Generalised USp(6) structure with constant singlet torsion
- $\mathbb{L}_K$ is action of $U(1)_R$ of dual SCFT

Exceptional Sasaki geometry

General solutions

Can we solve for the general supergravity solution dual to the deformed field theories? *Unlikely!*

- Solving for generic solutions seems intractable – no isometries; harder than Calabi–Yau

Instead, focus on holomorphic data for X

$$X \cdot K = 0, \qquad \mathbb{L}_K K = 0, \qquad \mu_+ = 0, \qquad \mathbb{L}_K X = 3\mathrm{i}X$$

- "Exceptional Sasaki" (ExS)
- X defines an exceptional complex structure [Tennyson, Waldram '21]

Moduli space

Let $\mathcal{Z}_K$ be the space of X structures which are ExS for a *fixed* K

$$\mathcal{Z}_K = \{X \mid \mu_+ = 0,\, \mathbb{L}_K K = 0,\, X \cdot K = 0,\, \mathbb{L}_K X = 3\mathrm{i}X\}$$

Final SUSY condition is a moment map for GDiff_K

$$\mu_K \coloneqq \mu_3 - 3 \int c(K, K, \,\cdot\,)$$

with moduli space

$$\mathcal{M}_\mathrm{c} \simeq \frac{\{X \in \mathcal{Z}_K \mid \mu_K = 0\}}{\mathrm{GDiff}_K} \equiv \mathcal{Z}_K \,/\!/\, \mathrm{GDiff}_K$$

- $\mathcal{M}_\mathrm{c}$ is conformal manifold of dual SCFT [Kol '02, Kol '10, Green et al. '10]

GIT quotient

$\mathcal{Z}_K$ admits a GDiff_K-invariant Kähler structure

Symplectic quotient equivalent to GIT quotient by complexified action

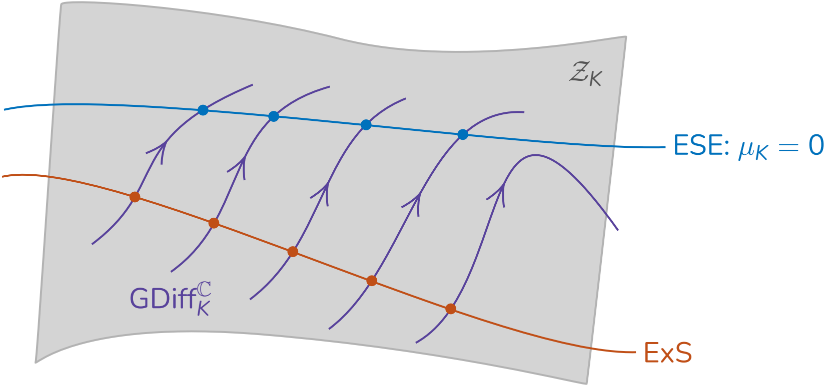
$$\mathcal{M}_\mathrm{c} \simeq \mathcal{Z}_K \,/\!/\, \mathrm{GDiff}_K \simeq \mathcal{Z}_K \,/\, \mathrm{GDiff}_K^{\mathbb{C}}$$

Final supersymmetry condition imposed by quotient

- In favourable case, given ExS structure (X, K), orbit of X under $\mathrm{GDiff}_K^{\mathbb{C}}$ intersects with $\mu_K = 0$

Useful: find simpler ExS structure for complicated geometry

Stability and existence



Subtlety: only the subset of polystable points have orbits that reach $\mu_K = 0$

Interpretation

X and K encode supergravity hyper- and vector-multiplet degrees of freedom

- Hypers dual to chiral multiplets in field theory

Holomorphic data of $\mathcal{W}$ encoded by X up to $\mathrm{GDiff}^{\mathbb{C}}$

$$[X] = \{X' = \Phi^*(X) \mid \mu_+ = 0,\ \Phi \in \mathrm{GDiff}^{\mathbb{C}}\}$$

Full solution (X, K) may be out of reach, but can solve for a simpler representative of $[X]$

- Any field theory quantity determined by $\mathcal{W}$, e.g. chiral spectrum, can be computed using any representative of $[X]$

Explicit ExS solutions for deformed geometries

Strategy

1. Start with known solution (X, K)
2. Deform to (X_{ED}, K) which solves weaker ExS conditions
3. Argue that X_{ED} can be $\text{GDiff}_K^{\mathbb{C}}$-transformed to solve $\mu_K = 0$, i.e. a full supersymmetric solution
4. Use (X_{ED}, K) to calculate interesting quantities characterising the dual deformed field theory

This is a completely general picture, but we'll focus on cases where the known solution is Sasaki–Einstein

Deformations of Sasaki–Einstein

New exceptional Sasaki solution for deformed SE geometries

$$K = K_{\mathrm{SE}}$$
$$X_{\mathrm{ED}}(f) = \mathrm{e}^{-\frac{1}{4}\mathrm{i}\,\Omega\wedge\bar{\Omega}}\mathrm{e}^{r^i A(f)}\mathrm{e}^{r^i r_j \varepsilon(f)}\left(s^i(f)\,\sigma\wedge\Omega + \mathrm{d}f\right)$$

where

- f is holomorphic and charge three, $\mathcal{L}_\xi f = 3\mathrm{i}f$
- $\mathrm{d}f \sim \beta \lrcorner (\sigma\wedge\Omega)$ with bivector $\beta \sim (\sigma\wedge\bar{\Omega})^\sharp \lrcorner \mathrm{d}f$
- Two-form $A(f)$ linear in f
- Function $\varepsilon(f)$ quadratic in f
- K_{SE} is unchanged

Geometry satisfies all susy conditions except $\mu_K = 0$

Are these stable?

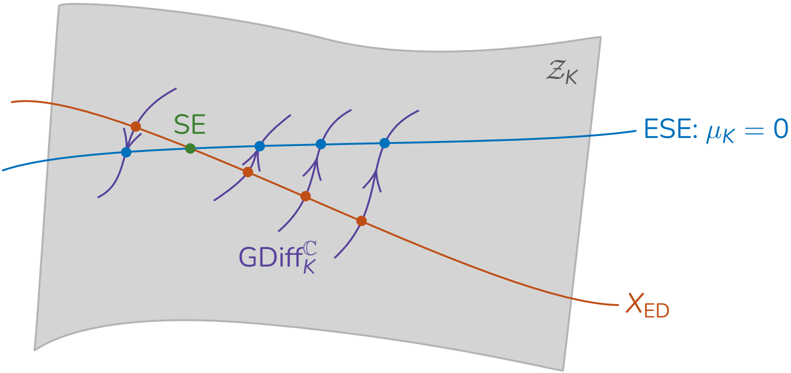
Given these solutions to ExS conditions, we know:

1. There is an open subset of stable points around $X_{\text{ED}}(0) = X_{\text{SE}}$
2. $X_{\text{ED}}(f)$ gives continuous one-parameter family of solutions under $f \mapsto \lambda f$
3. Linearised $X_{\text{ED}}(f)$ matches known infinitesimal solutions to $\mu_K = 0$ that necessarily lie in a stable subset [AA, Gabella, Graña, Petrini, Waldram '16]

Implies that for small but finite f, all $X_{\text{ED}}(f)$ are stable and can be completed to full supersymmetric solutions

- Existence of the deformed supergravity backgrounds

Physical interpretation



- Orbit $[X_{\mathrm{ED}}] \simeq \mathrm{GDiff}_K^{\mathbb{C}} \cdot X_{\mathrm{ED}}$ fixes superpotential $\mathcal{W}$ of dual field theory
- $\mathbb{L}_K X = 3iX$ fixes $\delta\mathcal{W}$ to be a marginal deformation (dual theory is conformal)
- Motion along orbit $\equiv$ renormalisation of Kähler potential

Example: deformations of five-sphere

Holomorphic charge-k functions descend from cone $C(S^5) = \mathbb{C}^3$

$$f = f_{i_1 \dots i_k} z^{i_1} \dots z^{i_k}, \qquad \mathcal{L}_\xi f = \mathrm{i}kf$$

(X_{ED}, K) is exceptional Sasaki for any $\mathcal{L}_\xi f = 3\mathrm{i}f$

- Explicit expressions for $A(f)$ and $\varepsilon(f)$
- Reproduces second-order perturbative analysis of [Aharony, Kol, Yankielowicz '02]
- For $f = z^1 z^2 z^3$, can find the $\mathrm{GDiff}_K^{\mathbb{C}}$ that transforms X_{ED} to known β-deformed solution [Lunin, Maldacena '05]

Same analysis for deformations of any Sasaki–Einstein background, e.g. $T^{1,1}$, etc.

Further deformations and counting chiral operators

Chiral spectrum

What can we calculate using the ExS solutions? $[X_{\mathrm{ED}}]$ fixes superpotential so should encode chiral spectrum of dual field theory

$$\text{chirals} = \{\text{chiral operators}\} \,/\, \{\mathrm{d}\mathcal{W} = 0\}$$

Can count these graded by R-charge $\rightarrow$ Hilbert series

- Counting for Sasaki–Einstein done by [Eager, Schmude, Tachikawa '12]
- But we want to count for the deformed theory! Hard as $\mathrm{d}\mathcal{W} = 0$ defines a non-commutative algebra

Chiral spectrum

Equivalent to further deformations of X with no constraint on charge, $\mathbb{L}_K X \neq 3\mathrm{i}X$

$$\text{chirals} = \{\delta X \mid \delta X \cdot K = 0,\ \delta\mu_+ = 0\} \,/\, \mathrm{GDiff}_K^{\mathbb{C}}$$

- Deformations of SCFT preserving $N = 1$ but not conformal

Counting δX up to symmetries defines a cohomology since

$$E_{\mathbb{C}} \xrightarrow{\mathbb{L}_\bullet X} T\mathcal{Z}_K \xrightarrow{\delta\mu_+} E_{\mathbb{C}}^*$$

Counting depends only on class $[X]$ and can be graded by charge under $\mathbb{L}_K$

Example: five-sphere

Chiral spectrum around $N = 4$ theory $\equiv$ charge-k deformations of X_{SE}, i.e. $X_{\text{ED}}(f)$ *without* restricting to $k = 3$

$$f = f_{i_1 \dots i_k} z^{i_1} \dots z^{i_k}, \qquad \mathcal{L}_\xi f = ikf$$

charge-k deformations $n_k \equiv$ # symmetric polynomials in (z^1, z^2, z^3)

$$H(t) = \sum_k n_k t^k = \frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \dots$$

- Matches Hilbert series of $N = 4$ SYM theory ✓
- General SE case counted by Kohn–Rossi cohomology

  [Eager, Schmude, Tachikawa '12]

Chiral spectrum at X_{ED}

When deformed solution is generic ($\eta \coloneqq \mathrm{d}f \neq 0$, “type one”)

$$[X_{\text{ED}}] \simeq \mathrm{e}^{b^i(f)+c_4(f)}\eta$$

cohomology reduces to “η cohomology” [Tasker ‘21]

$$\ldots \xrightarrow{\mathrm{d}} \eta \wedge \Lambda^p T^* \xrightarrow{\mathrm{d}} \eta \wedge \Lambda^{p+1} T^* \xrightarrow{\mathrm{d}} \ldots$$

fixed by Kohn–Rossi cohomology of original Sasaki–Einstein

Result: universal expression for Hilbert series

$$\boxed{H(t) \equiv \sum_k n_k t^k = 1 + \mathcal{I}_{\text{s.t.}}(t) - [k \equiv_3 0, k > 0] t^{2k}}$$

in terms of “single-trace superconformal index” $\mathcal{I}_{\text{s.t.}}(t)$

Example: deformed five-sphere

e.g. deformed S^5 with

$$f = f_\beta z^1 z^2 z^3 + f_\lambda \left[(z^1)^3 + (z^2)^3 + (z^3)^3 \right]$$

Hilbert series is

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + \dots$$

in agreement with counting from cyclic homology of Sklyanin algebra [Van den Bergh '94] ✓

New results for deformations of regular SEs: $T^{1,1}$, $\#n(S^2 \times S^3)$, etc.

Summary

Background geometry naturally encodes superpotential of dual SCFT

Can find supergravity solution for deformations up to $\mathrm{GDiff}_K^{\mathbb{C}}$ action – large class of new supergravity duals

Class of structure $[X]$ determines spectrum of chiral operators

Outlook

- Same/similar formalism for $\mathrm{AdS}_5/\mathrm{AdS}_4$ in type II / M-theory
- New perspective on a-maximisation for supersymmetric flux backgrounds
- Cohomology gives supersymmetric index

Marginal deformations, holomorphic data & counting chirals

Moment maps

The μ_α are a triplet of moment maps for the action of

$$\mathrm{GDiff} \simeq \mathrm{diffeo} + \mathrm{gauge}$$

Infinitesimally, $V \in \Gamma(E) \simeq \mathfrak{gdiff}$ acts by

$$\delta J_\alpha = \mathbb{L}_V J_\alpha$$

Action preserves hyper-Kähler structure on space of J_α so that

$$\mu_\alpha(V) = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma} \int_M \mathrm{tr}(J_\beta \mathbb{L}_V J_\gamma)$$

Marginal vs exactly marginal deformations

The field theory result of [Kol '02, Kol '10, Green et al. '10] that all marginal deformations are exactly marginal unless there is a global symmetry follows directly from moment map structure

e.g. $\mathrm{AdS}_5 \times \mathrm{S}^5$, (X, K) preserved by SU(3)

- Linearised deformation parameterised by $f = f_{ijk} z^i z^j z^k$
- $\mu_\alpha(V)$ trivially zero for $V \in \mathrm{SU}(3)$
- Further moment map for SU(3) and quotient on $\{f_{ijk}\}$

$$\mu_{\mathrm{SU}(3)} \equiv f_{ikl}\bar{f}^{jkl} - \tfrac{1}{3}\delta_i^j f_{klm}\bar{f}^{klm} = 0$$

gives space of exactly marginal couplings

Flat X structures

X defines an (almost) exceptional complex structure via

[Tennyson, Waldram '21]

$$J \sim \frac{\mathrm{i}}{\mathrm{tr}(X\bar{X})}[X, \bar{X}]$$

which decomposes

$$E_{\mathbb{C}} \simeq L_1 \oplus L_{-1} \oplus L_0$$
$$\underline{27} \to \underline{6}_1 + \underline{6}_{-1} + \overline{\underline{15}}_0$$

Integrability

$$\text{ECS} \quad \Leftrightarrow \quad \mathbb{L}_V W \in \Gamma(L_1) \quad \forall\, V, W \in \Gamma(L_1)$$

Flat X structures

What is missing for $\mu_+ = 0$? Impose

$$\mathbb{L}_V X = 0 \quad \forall\, V \in \Gamma(L_{-1})$$

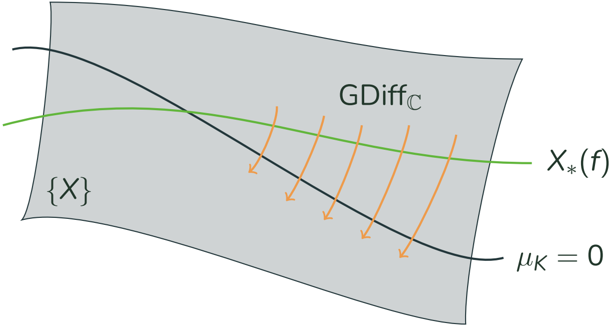
where $X\colon\ \det T \otimes E_{\mathbb{C}} \to L_{-1}$

General argument

Given solution (X_*, K) to ES conditions, can show that full solution exists:

1. Space of X with fixed K inherits invariant Kähler metric
2. $\mu_K(V) = \mu_3(V) - \int_M c(K, K, V)$ is moment map for GDiff with fixed K
3. (X_*, K) matches exactly marginal solutions for infinitesimal deformations
4. Open subset of stable points that lie on orbits of $\text{GDiff}_{\mathbb{C}}^K$ will intersect $\mu_K = 0$ – all (X_*, K) are stable and thus can be mapped to full solutions
5. Different X_* flow to different solutions unless there are isometries
6. X_* related by isometries map to same solution under $\text{GDiff}_{\mathbb{C}}^K$, in agreement with field theory [Kol '02, Kol '10, Green et al. '10]

Physical interpretation



1. Fixing an orbit $[X] \simeq \mathrm{GDiff}_{\mathbb{C}} \cdot X$ fixes the superpotential $\mathcal{W}$ of dual SCFT
2. $\mathbb{L}_K X = 3\mathrm{i}X$ fixes $\Delta = 3$ – marginal deformation
3. Motion along orbit $\equiv$ renormalisation of Kähler potential

Example: S^5 again

Mesonic operators

$\mathrm{tr}(\Phi\dots) \quad \leftrightarrow \quad$ holomorphic functions $f(z)$ on cone

- Marginal $\Rightarrow \mathcal{L}_\xi f = 3\mathrm{i}f$

Cone is $C(S^5) = \mathbb{C}^3$; functions are $f = f_{ijk}z^i z^j z^k$

Recall

$$X = \mathrm{e}^{\frac{1}{2}\mathrm{i}j^2} u^i\, \sigma \wedge \Omega \sim u^i\, \sigma \wedge \Omega \quad \text{up to GDiff}_{\mathbb{C}}$$

How do we deform this by f? Marginal for $\mathcal{L}_\xi f = 3\mathrm{i}f$

X_* for deformed S^5 background

New family of solutions to holomorphic conditions

$$K = \xi - \sigma \wedge j, \qquad X_* = e^{b^i(f)}(df + v^i(f)\, \sigma \wedge \Omega)$$

with $b^i \in \Lambda^2 T^*_{\mathbb{C}}$ linear and v^i quadratic in f

- In S^5 case and f cubic, reproduces second-order parts of [Aharony, Kol, Yankielowicz '02]
- If $f = z^1 z^2 z^3$, can solve for explicit $\mathrm{GDiff}_{\mathbb{C}}$ to take solution to exact β-deformed solution
- Works for deformation of any Sasaki–Einstein background – $T^{1,1}$, etc.

Chiral spectrum

What can we calculate using this (partial) solution?

X_* fixes superpotential so should encode space of mesonic operators, i.e. chiral ring

$$\text{chiral ring} =$$

$$\mathcal{O}_f = f_{ijkl\dots}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k\Phi^l\dots)$$

Can count these graded by R-charge $\rightarrow$ Hilbert series

- Counting for Sasaki–Einstein point known [Eager, Schmude, Tachikawa '12]
- But we want to count for the deformed theory!

Chiral spectrum

Counting δX up to $\mathrm{GDiff}_{\mathbb{C}}$ defines a cohomology since

$$E_{\mathbb{C}} \xrightarrow{\mathbb{L}_{\bullet} X} T\{X\} \xrightarrow{\delta\mu_+} E^*_{\mathbb{C}}$$

Cohomology counts chiral operators (drop $\mathbb{L}_K X = 3\mathrm{i}X$ condition)

$$\text{chirals} \sim \frac{\{\delta X \,|\, \delta\mu_+ = 0\}}{\{\delta X = \mathbb{L}_V X\}}$$

Counting depends only on class of X_* and $[X] = [X_*]$

Calculating the cohomoloy

Easiest when the deformed solution is generic – $\mathrm{d}f \neq 0$

- Using $\mathrm{GDiff}_{\mathbb{C}}$, can then write X_* as

$$X_* = \mathrm{e}^{\tilde{b}^i(\tau,f)+c_4(\tau,f)}\mathrm{d}f$$

Cohomology then reduces to [Tasker '21]

$$\dots \xrightarrow{\mathrm{d}} \mathrm{d}f \wedge \Lambda^p T^*_{\mathbb{C}} \xrightarrow{\mathrm{d}} \mathrm{d}f \wedge \Lambda^{p+1} T^*_{\mathbb{C}} \xrightarrow{\mathrm{d}} \dots$$

which can be computed using Kohn–Rossi cohomology of original Sasaki–Einstein

Counting chirals

Hilbert series

$$H(t) \equiv \sum_k n_k t^k = 1 + \mathcal{I}_{\text{s.t.}}(t) - [k \equiv_3 0, k > 0] t^{2k}$$

e.g. deformed S^5 with

$$f = f_\beta z^1 z^2 z^3 + f_\lambda \left[(z^1)^3 + (z^2)^3 + (z^3)^3 \right]$$

Hilbert series is

$$H(t) = \frac{(1+t)^3}{1-t^3} = 1 + 3t + 3t^2 + 2t^3 + \ldots$$

in agreement with [Van den Bergh '94]

New results

e.g. $T^{1,1}$ – undeformed result

$$H(t) = \frac{1 + t^{3/2}}{(1 - t^{3/2})^3} = 1 + 4t^{3/2} + 9t^3 + 16t^{9/2} \dots$$

For theory with generic deformed superpotential

$$H(t) = \frac{1 + 4t^{3/2} + 2t^3}{1 - t^3} = 1 + 4t^{3/2} + 3t^3 + 4t^{9/2} + \dots$$

- Matches explicit counting of gauge-invariant chiral field modulo F-term relations up to $k = 21/3$ [Tasker '21]
- No previous calculation of cyclic homology / chirals for deformed theory

New results for $\#n(S^2 \times S^3)$, etc.

Summary

Background geometry naturally encodes superpotential of dual SCFT

Can find supergravity solution for deformations up to $\mathrm{GDiff}_{\mathbb{C}}$ action – large class of new supergravity duals

Class of structure $[X]$ determines spectrum of chiral operators

Future

- Same/similar formalism for $\mathrm{AdS}_5/\mathrm{AdS}_4$ in M-theory
- Cohomology gives supersymmetric index
- a-maximisation for generic supersymmetric backgrounds – $a^{-1} \sim \int_M c(K, K, K)$